Diagonalizability over $\mathbb{R}$ and $\mathbb{C}$

Goal: Determine if matrix M is similar to a diagonal matrix.

IDEA: This will hold if and only if there is a basis for V ($= \mathbb{R}$ or $\mathbb{C}$) consisting of eigenvectors of M.

NB: When $M \in M_{n\times n}(\mathbb{R})$ has all eigenvalues real and M is diagonalizable, we say M diagonalizes over $\mathbb{R}$. When M has complex entries or eigenvalues, we must consider M as a complex matrix. In such cases (if M is still diagonalizable), we say that M diagonalizes over $\mathbb{C}$.

Algorithm (Compute $M = PDP^{-1}$ if it exists): Let M be a square matrix with possibly complex entries.

1. Compute $p_M(\lambda) = \det(M - \lambda I)$.
2. Solve $p_M(\lambda) = 0$ for eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.
3. For each distinct eigenvalue λ compute a basis $B_\lambda \subseteq V_\lambda$.
   ↳ if any geometric multiplicity is strictly less than the algebraic multiplicity of the same eigenvalue, STOP. This implies V does not have an 'eigenbasis' for M.
4. Let $E = \bigcup_{\lambda:\text{ e-val}} B_\lambda$. Then (if we passed step 3) the set E is a basis of V.
5. We have $M = PDP^{-1}$ for a diagonal matrix D and $P = \mathrm{Rep}_{E,A}(id)$.

$$\begin{tikzcd} V_A \arrow[r, "M"] \arrow[d, "P^{-1}"'] & V_A \\ V_E \arrow[r, "D"'] & V_E \arrow[u, "P"'] \end{tikzcd}$$

Specifically, if $E = \{v_1, v_2, \dots, v_n\}$ has associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ resp., then $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

← evil ?

Recall: If B and A are bases, then we compute $\mathrm{Rep}_{A,B}(id)$ via

$$\mathrm{RREF}\left[B \mid A\right] = \left[I \mid \mathrm{Rep}_{A,B}(id)\right].$$

The rest of these notes are copious examples...

Ex: We diagonalize $M = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

Char poly: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 1 & 1-\lambda \end{bmatrix}$

$= (2-\lambda)(1-\lambda) - 3 = \lambda^2 - 3\lambda - 1$

E-vals: $p_M(\lambda) = 0 \iff \lambda^2 - 3\lambda - 1 = 0 \iff \lambda = \dfrac{3 \pm \sqrt{9+4}}{2} = \frac{1}{2}(3 \pm \sqrt{13})$

E-spaces: Computing E-spaces separately:

$\lambda_1 = \frac{1}{2}(3 + \sqrt{5})$: $V_{\lambda_1} = \mathrm{null}(M - \lambda_1 I) = \mathrm{null}\begin{bmatrix} \frac{1}{2} - \frac{1}{2}\sqrt{13} & 3 \\ 1 & -\frac{1}{2} - \frac{1}{2}\sqrt{13} \end{bmatrix}$.

Now $\mathrm{RREF}\begin{bmatrix} \frac{1}{2} - \frac{1}{2}\sqrt{13} & 3 \\ 1 & -\frac{1}{2} - \frac{1}{2}\sqrt{13} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} - \frac{1}{2}\sqrt{13} \\ 0 & 0 \end{bmatrix}$ ← check!

So $\begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $x - \frac{1}{2}(1+\sqrt{13})y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+\sqrt{13})y \\ y \end{bmatrix} = \frac{1}{2}y\begin{bmatrix} 1+\sqrt{13} \\ 2 \end{bmatrix}$

Hence $B_{\lambda_1} = \left\{ \begin{bmatrix} 1+\sqrt{13} \\ 2 \end{bmatrix} \right\}$ is a basis of V_{λ_1}.

$\lambda_2 = \frac{1}{2}(3 - \sqrt{5})$: $V_{\lambda_2} = \mathrm{null}(M - \lambda_2 I) = \mathrm{null}\begin{bmatrix} \frac{1}{2} + \frac{1}{2}\sqrt{13} & 3 \\ 1 & -\frac{1}{2} + \frac{1}{2}\sqrt{13} \end{bmatrix}$

Now $\mathrm{RREF}\begin{bmatrix} \frac{1}{2} + \frac{1}{2}\sqrt{13} & 3 \\ 1 & -\frac{1}{2} + \frac{1}{2}\sqrt{13} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} + \frac{1}{2}\sqrt{13} \\ 0 & 0 \end{bmatrix}$ ← check!

Thus $\begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2}$ iff $x - \frac{1}{2}(1-\sqrt{13})y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1-\sqrt{13})y \\ y \end{bmatrix} = \frac{1}{2}y\begin{bmatrix} 1-\sqrt{13} \\ 2 \end{bmatrix}$

Hence $B_{\lambda_2} = \left\{ \begin{bmatrix} 1-\sqrt{13} \\ 2 \end{bmatrix} \right\}$ is a basis of V_{λ_2}.

Eigenbasis: Let $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1+\sqrt{13} \\ 2 \end{bmatrix}, \begin{bmatrix} 1-\sqrt{13} \\ 2 \end{bmatrix} \right\}$. Note $\# E = 2 = \dim(\mathbb{R}^2)$, and all e-values and entries of M are real. Hence M is diagonalizable over $\mathbb{R}$:)

Now $\mathrm{Rep}_{E,\mathcal{E}_2}(id) = [E] = \begin{bmatrix} 1+\sqrt{13} & 1-\sqrt{13} \\ 2 & 2 \end{bmatrix}$ and

$$\mathrm{Rep}_{\mathcal{E}_2,E}(id) = \mathrm{Rep}_{E,\mathcal{E}_2}(id)^{-1} = \frac{1}{2(1+\sqrt{13})-2(1-\sqrt{13})}\begin{bmatrix} 2 & -(1-\sqrt{13}) \\ -2 & 1+\sqrt{13} \end{bmatrix}$$

2×2 matrix inverse formula

$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} 2 & -1+\sqrt{13} \\ -2 & 1+\sqrt{13} \end{bmatrix}$$

Hence we may write:

$$M = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(L) = \mathrm{Rep}_{\mathcal{E}_2,E}(id)\,\mathrm{Rep}_{E,E}(L)\,\mathrm{Rep}_{E,\mathcal{E}_2}(id) = PDP^{-1}$$

i.e. $$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{13} & 1-\sqrt{13} \\ 2 & 2 \end{bmatrix} D \cdot \frac{1}{4\sqrt{13}}\begin{bmatrix} 2 & -1+\sqrt{13} \\ -2 & 1+\sqrt{13} \end{bmatrix}$$

NB: At this point we expect $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3+\sqrt{13}) & 0 \\ 0 & \frac{1}{2}(3-\sqrt{13}) \end{bmatrix}$

Let's check that!

Check: multiplying on the left by P^{-1} and on the right by P:

$$D = \frac{1}{4\sqrt{13}}\begin{bmatrix} 2 & -1+\sqrt{13} \\ -2 & 1+\sqrt{13} \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1+\sqrt{13} & 1-\sqrt{13} \\ 2 & 2 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} 4-1+\sqrt{13} & 6-1+\sqrt{13} \\ -4+1+\sqrt{13} & -6+1+\sqrt{13} \end{bmatrix}\begin{bmatrix} 1+\sqrt{13} & 1-\sqrt{13} \\ 2 & 2 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} 3+\sqrt{13} & 5+\sqrt{13} \\ -3+\sqrt{13} & -5+\sqrt{13} \end{bmatrix}\begin{bmatrix} 1+\sqrt{13} & 1-\sqrt{13} \\ 2 & 2 \end{bmatrix}$$

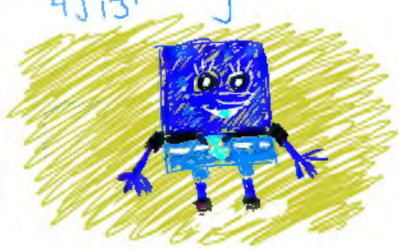
$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} (3+\sqrt{13})(1+\sqrt{13})+2(5+\sqrt{13}) & (3+\sqrt{13})(1-\sqrt{13})+2(5+\sqrt{13}) \\ (-3+\sqrt{13})(1+\sqrt{13})+2(-5+\sqrt{13}) & (-3+\sqrt{13})(1-\sqrt{13})+2(-5+\sqrt{13}) \end{bmatrix}$$

$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} 3+4\sqrt{13}+13+10+2\sqrt{13} & 3-2\sqrt{13}-13+10+2\sqrt{13} \\ -3-2\sqrt{13}+13-10+2\sqrt{13} & -3+4\sqrt{13}-13-10+2\sqrt{13} \end{bmatrix}$$

$$= \frac{1}{4\sqrt{13}}\begin{bmatrix} 26+6\sqrt{13} & 0 \\ 0 & -26+6\sqrt{13} \end{bmatrix} = \begin{bmatrix} \frac{2(13+3\sqrt{13})}{4\sqrt{13}} & 0 \\ 0 & \frac{2(-13+3\sqrt{13})}{4\sqrt{13}} \end{bmatrix}$$

Bizarro Sponge...

$$= \begin{bmatrix} \frac{1}{2}(\sqrt{13}+3) & 0 \\ 0 & \frac{1}{2}(-\sqrt{13}+3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3+\sqrt{13}) & 0 \\ 0 & \frac{1}{2}(3-\sqrt{13}) \end{bmatrix}$$



Ex: We diagonalize $M = \begin{bmatrix} -9 & -4 \\ 24 & 11 \end{bmatrix}$.

Char poly: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -9-\lambda & -4 \\ 24 & 11-\lambda \end{bmatrix}$

$$= (-9-\lambda)(11-\lambda) - 24(-4)$$
$$= -99 - 2\lambda + \lambda^2 + 96$$
$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

E-values: $p_M(\lambda) = 0$ iff $\lambda = 3$ or $\lambda = -1$

E-spaces: Analyzing our eigenvalues separately:

$\underline{\lambda_1 = -1}$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null}\begin{bmatrix} -9+1 & -4 \\ 24 & 11+1 \end{bmatrix} = \text{null}\begin{bmatrix} -8 & -4 \\ 24 & 12 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $2x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Hence $B_{\lambda_1} = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ is a basis of V_{λ_1}.

$\underline{\lambda_2 = 3}$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} -9-3 & -4 \\ 24 & 11-3 \end{bmatrix} = \text{null}\begin{bmatrix} -12 & -4 \\ 24 & 8 \end{bmatrix} = \text{null}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2}$ iff $3x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Hence $B_{\lambda_2} = \left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ is a basis of V_{λ_2}.

Eigenbasis: $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ has $\# E = 2 = \dim(\mathbb{R}^2)$, so

B is an eigenbasis for M; thus M diagonalizes over $\mathbb{R}$.

We can thus write $M = PDP^{-1}$ for some diagonal D and invertible P.

($\underline{\text{NB}}$: We know $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ because our basis E had eigenvalues $-1, 3$ resp)

Diagonalize: We recognize the matrix M as a transformation $\mathbb{R}^2 \xrightarrow{L_M} \mathbb{R}^2$. Thus

$$M = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(L) = \text{Rep}_{E,\mathcal{E}_2}(id)\ \text{Rep}_{E,E}(L)\ \text{Rep}_{\mathcal{E}_2,E}(id) = PDP^{-1}$$

$$\begin{tikzcd}
\mathbb{R}^2_{\mathcal{E}_2} \arrow[r, "\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(L)=M"] \arrow[d, "P^{-1} = \text{Rep}_{\mathcal{E}_2,E}(id)"'] & \mathbb{R}^2_{\mathcal{E}_2} \arrow[d, "\text{Rep}_{\mathcal{E}_2,E}(id)=P"] \\
\mathbb{R}^2_E \arrow[r, "\text{Rep}_{E,E}(L)=D"'] & \mathbb{R}^2_E
\end{tikzcd}$$

Now $P = \text{Rep}_{E,\mathcal{E}_2}(id) = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$

(b/c $[\mathcal{E}_2 | E]$ is in RREF :))

So $P^{-1} = \dfrac{1}{-3-(-2)}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$ (via 2x2 matrix inverse formula)

Check: we verify $PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} -9 & -4 \\ 24 & 8 \end{bmatrix} = M$:)

Not every matrix is diagonalizable over $\mathbb{R}$.

Ex: Let $M = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

Char Poly: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 1$

Eigenvalues: $p_M(\lambda) = 0$ iff $(2-\lambda)^2 + 1 = 0$ iff $\lambda = 2 \pm i$ $\leftarrow$ $\therefore$ not diagonalizable over $\mathbb{R}$

Eigenspaces: We analyze each eigenvalue separately.

$\lambda_1 = 2+i$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null}\begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix} = \text{null}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} = \text{null}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $x + iy = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ is a basis for V_{λ_1}.

$\lambda_2 = 2-i$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} 2-(2-i) & 1 \\ -1 & 2-(2-i) \end{bmatrix} = \text{null}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} = \text{null}\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2}$ iff $x - iy = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ is a basis for V_{λ_2}.

Eigenbasis: Hence $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ has $\#E = 2 = \dim(\mathbb{C}^2)$,

so M diagonalizes over $\mathbb{C}$; i.e. $M = PDP^{-1}$ for

$$P = \text{Rep}_{E,\mathcal{E}_2}(id) = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

Check: $P^{-1} = \frac{1}{-i-i}\begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} = \frac{1}{2}i\begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

$\uparrow \frac{1}{-i} = \frac{1 \cdot i}{-i \cdot i} = \frac{i}{1} = i$

Now $PDP^{-1} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1+2i & 2+i \\ -1-2i & 2-i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -i(-1+2i) + i(-1-2i) & -i(2+i) + i(2-i) \\ (-1+2i) + (-1-2i) & (2+i) + (2-i) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} i+2-i+2 & -2i+1+2i+1 \\ -1+2i-1-2i & 2+i+2-i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = M$$

Note: Even though this example didn't diagonalize over $\mathbb{R}$, it did diagonalize over $\mathbb{C}$.

Not every matrix diagonalizes (over $\mathbb{R}$ or $\mathbb{C}$).

Ex: Let $M = \begin{bmatrix} -1 & \pi \\ 0 & -1 \end{bmatrix}$. We attempt to diagonalize M.

Characteristic Polynomial: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -1-\lambda & \pi \\ 0 & -1-\lambda \end{bmatrix} = (-1-\lambda)^2$

Eigenvalues: $p_M(\lambda) = 0$ iff $(-1-\lambda)^2 = 0$ iff $\lambda = -1$

Eigenspace: When $\lambda = -1$ we see $V_\lambda = \text{null}(M - \lambda I) = \text{null}\begin{bmatrix} 0 & \pi \\ 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_\lambda$ iff $y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Hence $B_\lambda = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis for V_λ.

Note the algebraic multiplicity of λ is 2, while the geometric multiplicity of λ is only 1. Hence $\mathbb{R}^2$ does not have a basis of eigenvectors of M. In particular, M is not diagonalizable (over $\mathbb{R}$ or $\mathbb{C}$)!

Ex: Diagonalize $M = \begin{bmatrix} -4 & 1 \\ -1 & -6 \end{bmatrix}$ if possible.

Sol: Characteristic Poly: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -4-\lambda & 1 \\ -1 & -6-\lambda \end{bmatrix}$

$$= (-4-\lambda)(-6-\lambda) - (-1 \cdot 1)$$

$$= \lambda^2 + 10\lambda + 24 + 1 = (\lambda+5)^2$$

Eigenvalues: $p_M(\lambda) = 0$ iff $(\lambda+5)^2 = 0$ iff $\lambda = -5$.

Eigenspace: When $\lambda = 5$, note $V_\lambda = \text{null}(M - \lambda I) = \text{null}\begin{bmatrix} -4-(-5) & 1 \\ -1 & -6-(-5) \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus $\begin{bmatrix} x \\ y \end{bmatrix} \in V_\lambda$ iff $x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $B_\lambda = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is a basis of V_λ.

Because $\dim(V_\lambda) = 1 < 2 = \text{alg. mult of } \lambda$, we see M is not diagonalizable.

Ex: Diagonalize $\begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix}$ if possible.

Sol: characteristic poly: $p_M(\lambda) = \det\begin{bmatrix} -\lambda & 2 \\ 8 & -\lambda \end{bmatrix} = \lambda^2 - 16 = (\lambda-4)(\lambda+4)$ E-vals: $\lambda = \pm 4$.

$\lambda = -4$: $V_\lambda = \text{null}\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_\lambda$ iff $2x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\therefore B_\lambda = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ is a basis of V_λ.

$\lambda = 4$: $V_\lambda = \text{null}\begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix} = \text{null}\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_\lambda$ iff $-2x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\therefore B_\lambda = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is a basis of V_λ.

Diagonalize: Hence $\begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1}$ Check!

(NB: that's just $M = PDP^{-1}$ ☺)

Ex: Diagonalize $M = \begin{bmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{bmatrix}$ if possible.

Sol: We apply our diagonalization algorithm

Char poly: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} -5-\lambda & 0 & 6 \\ -3 & 1-\lambda & 3 \\ -3 & 0 & 4-\lambda \end{bmatrix}$ cofactor expand here.

$$= (1-\lambda) \det \begin{bmatrix} -5-\lambda & 6 \\ -3 & 4-\lambda \end{bmatrix} = (1-\lambda)\left((-5-\lambda)(4-\lambda) - 6(-3)\right)$$

$$= (1-\lambda)(-20 + \lambda + \lambda^2 + 18) = (1-\lambda)(\lambda^2 + \lambda - 2)$$

$$= (1-\lambda)(\lambda+2)(\lambda-1) = (1-\lambda)^2(-2-\lambda)$$

Eigenvalues: $p_M(\lambda) = 0$ iff $\lambda = 1$ or $\lambda = -2$, so $\lambda_1 = 1$, $\lambda_2 = -2$.

Eigenspaces: Separate computation by eigenvalue.

$\underline{\lambda_1 = 1}$: $V_{\lambda_1} = \operatorname{null}(M - \lambda_1 I) = \operatorname{null} \begin{bmatrix} -5-1 & 0 & 6 \\ -3 & 1-1 & 3 \\ -3 & 0 & 4-1 \end{bmatrix} = \operatorname{null} \begin{bmatrix} -6 & 0 & 6 \\ -3 & 0 & 3 \\ -3 & 0 & 3 \end{bmatrix} = \operatorname{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $x - z = 0$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\therefore$ Basis $B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\underline{\lambda_2 = -2}$: $V_{\lambda_2} = \operatorname{null}(M - \lambda_2 I) = \operatorname{null} \begin{bmatrix} -5+2 & 0 & 6 \\ -3 & 1+2 & 3 \\ -3 & 0 & 4+2 \end{bmatrix} = \operatorname{null} \begin{bmatrix} -3 & 0 & 6 \\ -3 & 3 & 3 \\ -3 & 0 & 6 \end{bmatrix}$

$$= \operatorname{null} \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{null} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_2}$ iff $\begin{cases} x - 2z = 0 \\ -y + z = 0 \end{cases}$ iff $\begin{cases} x = 2t \\ y = t \\ z = t \end{cases}$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. $\therefore$ Basis $B_{\lambda_2} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Eigenbasis: Hence $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Finally, we verify $M = PDP^{-1}$ for $P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$:

To compute P^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$

$$\therefore PDP^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0-4 & 0+0+0 & 2+0+4 \\ 0-1-2 & 0+1+0 & 0+1+2 \\ -1-0-2 & 0+0+0 & 2+0+2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{bmatrix} = M$$

Ex: Diagonalize $M = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 3 & 3 \end{bmatrix}$ if possible.

Sol: Char poly: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 & -1 \\ 2 & -2-\lambda & -2 \\ -1 & 3 & 3-\lambda \end{bmatrix}$

$$= (3-\lambda)\det\begin{bmatrix} -2-\lambda & -2 \\ 3 & 3-\lambda \end{bmatrix} - (-1)\det\begin{bmatrix} 2 & -2 \\ -1 & 3-\lambda \end{bmatrix} + (-1)\det\begin{bmatrix} 2 & -2-\lambda \\ -1 & 3 \end{bmatrix}$$

$$= (3-\lambda)\big((-2-\lambda)(3-\lambda) - (-2)3\big) + \big(2(3-\lambda) - (-1)(-2)\big) - \big(2\cdot 3 - (-1)(-2-\lambda)\big)$$

$$= (3-\lambda)(-6 - \lambda + \lambda^2 + 6) + (6 - 2\lambda - 2) - (6 + 2 - \lambda)$$

$$= (3-\lambda)(\lambda^2 - \lambda) + (4 - 2\lambda) + (-4 + \lambda)$$

$$= \lambda(3-\lambda)(\lambda - 1) - \lambda = \lambda(3\lambda - 3 - \lambda^2 + \lambda - 1)$$

$$= \lambda(-\lambda^2 + 4\lambda - 4) = -\lambda(\lambda^2 - 4\lambda + 4) = -\lambda(2-\lambda)^2$$

Hence we have eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$.

$\lambda_1 = 0$ Eigenspace: $V_{\lambda_1} = \operatorname{null}(M - \lambda_1 I) = \operatorname{null}\begin{bmatrix} 3 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 3 & 3 \end{bmatrix} = \operatorname{null}\begin{bmatrix} 1 & -3 & -3 \\ 2 & -2 & -2 \\ 3 & -1 & -1 \end{bmatrix}$

$$= \operatorname{null}\begin{bmatrix} 1 & -3 & -3 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} = \operatorname{null}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $\begin{cases} x = 0 \\ y + z = 0 \end{cases}$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. $\therefore$ Basis $\mathcal{B}_{\lambda_1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

$\lambda_2 = 2$ Eigenspace: $V_{\lambda_2} = \operatorname{null}(M - \lambda_2 I) = \operatorname{null}\begin{bmatrix} 1 & -1 & -1 \\ 2 & -4 & -2 \\ -1 & 3 & 1 \end{bmatrix} = \operatorname{null}\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \operatorname{null}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_2}$ iff $\begin{cases} x + z = 0 \\ y = 0 \end{cases}$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. $\therefore$ Basis $\mathcal{B}_{\lambda_2} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Hence the geometric multiplicity of λ_2 is strictly less than its algebraic multiplicity, so M is not diagonalizable.